மனோன்மணியம் சுந்தரனார் பல்கலைக்கழகம்

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தொலைநிலை தொடர்கல்வி இயக்ககம்

**DIRECTORATE OF DISTANCE AND**

**CONTINUING EDUCATION**

# B.Sc. MATHEMATICS

## I YEAR

# DIFFERENTIAL CALCULUS

**Sub. Code: JMMA12** **(For Private Circulation only)**

# B.Sc. MATHEMATICS –I YEAR

## JMMA12: DIFFERENTIAL CALCULUS

## SYLLABUS

**Unit-1**

**Successive Differentiation**: Introduction (Review of basic concepts) – The $n^{th}$ derivative – Standard results – Trigonometrical transformation – Formation of equationsinvolving derivatives – Leibnitz formula for the $n^{th}$ derivative of a product.

**Unit-2**

**Partial Differentiation**: Partial derivatives – Successive partial derivatives –Function of a function rule – Total differential coefficient.

**Unit-3**

**Partial Differentiation (Continued)**: Homogeneous functions – Partial derivatives of a function of two variables - Lagrange’s method of undetermined multipliers.

**Unit-4**

**Envelope:** Method of finding the envelope – Another definition of envelope – Envelope of family of curves which are quadratic in the parameter.

**Unit-5**

**Curvature:** Definition of Curvature – Circle, Radius and Centre of Curvature –Evolutes and Involutes – Radius of Curvature in Polar Co-ordinates.

**Text Book**


1. S. Narayanan and T.K. Manicavachagam Pillay, Calculus-
   Volume I, S.Viswananthan Printers & Publication Pvt. Ltd. 2015
2. S. Arumugam and A. Thangapandi Issac, Calculus,
   New Gamma Publishing House, Palayamkottai 2011

# JMMA12: DIFFERENTIAL CALCULUS

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**Unit-1 SUCCESSIVE DIFFERENTIATION**:

Introduction (Review of basic concepts) – The $n^{th}$ derivative – Standard results – Trigonometrical transformation – Formation of equationsinvolving derivatives – Leibnitz formula for the $n^{th}$ derivative of a product.

**SUCCESSIVE DIFFERENTIATION**

**1.1 Introduction:**

We have seen that the derivative of a function of X is also a function of x. The new function may be differentiable, in which case, the derivative of the first derivative is called the second derivative of the original function. Similarly the derivative of the second derivative is called the third derivative, and so on up to the $n^{th}$ derivative.

Thus if $y = 4x^5$

$$\frac{dy}{dx} = 20x^4$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 80x^3$$

$$\frac{d}{dx}\left\{\frac{d}{dx}\left(\frac{dy}{dx}\right)\right\} = 240x^2 \text{ , etc.}$$

The symbols of the successive derivate are usually abbreviated as follows:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D^2y$$

$$\frac{d}{dx}\left\{\frac{d}{dx}\left(\frac{dy}{dx}\right)\right\} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = D^3y$$

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = D^ny$$

If $y = f(x)$, the successive derivatives are also denoted by

$f'(x), f''(x), \dots \dots . f^n(x),$

$y', y'', \dots \dots \dots . y^{(n)},$

$y_1, y_2 \dots \dots y_n$

**1.2.The $n^{th}$ derivative:**

For certain functions a general expression involving n may be found for the $n^{th}$ derivative. The usual plan is to find number of successive derivatives, as many as be necessary to discover their law of formation and then by induction write down the $n^{th}$ derivative.

For example, $y = e^{ax}$

$$\frac{dy}{dx} = ae^{ax}$$

$$\frac{d^2y}{dx^2} = a^2e^{ax}$$

Then $\frac{d^ny}{dx^n} = a^ne^{ax}$

**Standard Results**

1. If $y = (ax + b)^m$ , then

$$y_1 = ma(ax + b)^{m-1}$$

$$y_2 = m(m - 1)a^2(ax = b)^{m-2}$$

.....

$$y_n = m(m - 1) \dots \dots (m - n + 1)a^n(ax + b)^{m-n}$$

In particular, $D^n(ax + b)^{-1} = (-1)^n n!\, a^n(ax + b)^{n-1}$

2. If $y = \log(ax + b)$

$$y_1 = a(ax + b)^{-1}$$

........

$$y_n = a\frac{d^{n-1}}{dx^{n-1}}(ax + b)^{-1}$$

$$= a(-1)^{n-1}(n - 1)!\, a^{n-1}(ax + b)^{-n}$$

$$= (-1)^{n-1}(n - 1)!\, a^n(ax + b)^{-n}$$

3. If $n^{th}$ derivative of $\sin(ax+b)$

Let $y = \sin(ax+b)$

$y_1 = \mathrm{a}\cos(ax+b)$

Thus the effect of a differentiation is to multiply by a and increase the

angle by $\frac{\pi}{2}$

$y_1 = \mathrm{a}\sin(\frac{\pi}{2} + ax + b)$

$y_2 = \mathrm{a}\cos\,(\frac{\pi}{2} + ax + b) = \mathrm{a}^2\sin(\frac{2\pi}{2} + ax + b)$

$y_3 = \mathrm{a}^3\sin(\frac{3\pi}{2} + ax + b)$

In general, $D^n \sin(ax+b) = \mathrm{a}^{\mathrm{n}}\sin(\frac{n\pi}{2} + ax + b)$

Similarly $D^n \sin(ax+b) = \mathrm{a}^{\mathrm{n}}\cos(\frac{n\pi}{2} + ax + b)$

4. Find the $n^{th}$ derivative of $e^{ax}\sin(bx+c)$

Let $y = e^{ax}\sin(bx+c)$

$y_1 = e^{ax}bcos(bx+c) + ae^{ax}\sin(bx+c)$

Putting $a = rcos\emptyset$ and $b = rsin\varphi$

We have, $y_1 = re^{ax}\sin(bx+c+\varphi)$

Thus the effect of a differentiation is to multiply by r and increase the angle by $\varphi$

Similarly, $y_2 = r^2 e^{ax}\sin(bx+c+2\varphi)$

In general, $D^n\{e^{ax}\sin(bx+c)\} = r^n e^{ax}\sin(bx+c+n\varphi)$

Where $r = (a^2+b^2)^{1/2}$ and $\varphi = \tan^{-1} b/a$

Similarly, $D^n\{e^{ax}\cos(bx+c)\} = r^n e^{ax}\cos(bx+c+n\varphi)$

Fractional expressions of the form $\frac{f(x)}{\varphi(x)}$ , both functions being algebraic and rational, can be differentiated n times by splitting them into partial fractions.

**Example 1:** Find $y_n$, where $y = \frac{3}{(x+1)(2x-1)}$ in partial fraction

**Solution:**

Let $\frac{3}{(x+1)(2x-1)} = \frac{A}{2x-1} + \frac{B}{x+1} \ldots \ldots .. (1)$

$$\frac{3}{(x+1)(2x-1)} = \frac{A(x+1)+B(2x-1)}{(2x-1)(x+1)}$$

$$3 = A(x+1) + B(2x-1)$$

Put $x = -1 \implies 3 = A(-1+1) + B(-3)$

$$3 = -3B$$

$$B = -1$$

Put $x = 2 \implies 3 = A(2+1) + B(4-1)$

$$3 = 3A - 3 \implies A = 2$$

Sub in equation (1)

$$y = \frac{2}{2x-1} - \frac{1}{x+1}$$

$D^n(ax-b)^{-1} = (-1)^n n!\, a^n (ax+b)^{-n-1}$

$$\frac{2}{2x-1} = \frac{2(-1)^n n! 2^n}{(2x-1)^{n+1}}$$

$$\frac{1}{x+1} = \frac{(-1)^n n!}{(x+1)^{n+1}}$$

$$y = \frac{2(-1)^n n!\, 2^n}{(2x-1)^{n+1}} - \frac{(-1)^n n!}{(x+1)^{n+1}}$$

$$y = (-1)^n n! \{\frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{1}{(x+1)^{n+1}}\}$$

**Example 2:** Find $y_n$, where $y = \frac{x^2}{(x-1)^2(x+2)}$

**Solution:**

Let $\frac{x^2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$ …….. (1)

multiply both sides by $(x-1)^2(x+2)$, we get

$x^2 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$ …….. (2)

Put $x = -2 \Longrightarrow (-2)^2 = A(-2-1)(-2+2) + B(-2+2) + C(-2-1)^2$

$$C = \frac{4}{9}$$

Put $x = -1$

$$(-1)^2 = A(-1-1)(-1+2) + B(-1+2) + C(-1-1)^2$$

$$1 = -2A + B + 4\left(\frac{4}{9}\right)$$

$$2A = \frac{1}{3} + \frac{16}{9} - 1$$

$$A = \frac{5}{9}$$

Sub in equation (1), $y = \frac{5}{9(x-1)} + \frac{1}{3(x-1)^2} + \frac{4}{9(x+2)}$

$$\frac{5}{9(x-1)} = \frac{5}{9}\frac{(-1)^n n!}{(x-1)^{n+1}}$$

$$\frac{1}{3(x-1)^2} = \frac{1}{3}\frac{(n+1)!(-1)^n}{(x-1)^{n+2}}$$

$$\frac{4}{9(x+2)} = \frac{4}{9}\frac{(-1)^n n!}{(x+2)^{n+1}}$$

$$y = \frac{5}{9}\frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{1}{3}\frac{(n+1)!(-1)^n}{(x-1)^{n+2}} + \frac{4}{9}\frac{(-1)^n n!}{(x+2)^{n+1}}$$

$$y = (-1)^n n!\left[\frac{5}{9}\frac{1}{(x-1)^{n+1}} + \frac{1}{3}\frac{(n+1)}{(x-1)^{n+2}} + \frac{4}{9}\frac{1}{(x+2)^{n+1}}\right]$$

**Example 3:** Find $y_n$, where $y = \frac{1}{x^2+a^2}$

**Solution:**

$$y = \frac{1}{x^2+a^2} = \frac{1}{2ai}\left[\frac{1}{x-ai} - \frac{1}{x+ai}\right]$$

$$y_n = \frac{(-1)^n n!}{2ai}\left[\frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}}\right]$$

**Exercise 1:**

1.Find $y_n$ when,

(a) $y = tan^{-1}\frac{x}{a}$

(b) $y = \frac{1}{(x+a)^2+b^2}$

(c) $y = \frac{1}{(x^2+a^2)(x^2+b^2)}$

(d) $y = \frac{x}{(x-1)^2(x+2)}$

**1.3. Trigonometrical Transformation**

It is possible to break up products of powers of sines and cosines into a sum by trigonometrical methods.

**Example 1:**Find the $n^{th}$ differential coefficient of $cosx\ cos2x\ cos3x$

**Solution:**

$y = cosx \cdot cos2x \cdot cos3x$

$$y = cosxcos2xcos3x = \frac{1}{2}cosx[cos5x + cosx]$$

$$y = \frac{1}{2}(cosx \cdot cos5x + cosx \cdot cosx)$$

$$\therefore y = \frac{1}{4}[cos6x + cos4x + 1 + cos2x]$$

Usign formula,

$$y = cos(ax+b) then\ D^n \cos(ax+b) = a^n cos(ax + b + \frac{n\pi}{2})$$

$$\therefore D^n(cosx \cdot cos2x \cdot cos3x) = \frac{1}{4}[6^n cos(6x + \frac{n\pi}{2}) + 4^n cos(4x + \frac{n\pi}{2}) + 2^n cos(2x + \frac{n\pi}{2})]$$

**Example 2:** Find the $n^{th}$ differential coefficient of $\cos^5\theta\sin^7\theta$

**Solution:**

Let x = cos θ + i sin θ

$\frac{1}{x} = \cos\theta - i\sin\theta$

then $x + \frac{1}{x} = 2\cos\theta$

Now $\Rightarrow \cos\theta = \frac{1}{2}(x + \frac{1}{x})$ …………(1)

Similarly $x - \frac{1}{x} = 2\text{i}\sin\theta$

Now $\Rightarrow \sin\theta = \frac{1}{2i}(x - \frac{1}{x})$ …………(2)

By De Moivre's theorem, $x^n = \cos n\theta + i\sin n\theta$

then $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

Now, $x^n + \frac{1}{x^n} = 2\cos n\theta$

Similarly, $x^n - \frac{1}{x^n} = 2i\sin n\theta$

From equation (1), $\cos^5\theta$ $= \left[\frac{1}{2}(x + \frac{1}{x})\right]^5 = \frac{1}{2^5}(x + \frac{1}{x})^5$

From equation (2), $\sin^7\theta$ $= \left[\frac{1}{2i}(x - \frac{1}{x})\right]^7 = \frac{1}{(2i)^7}(x - \frac{1}{x})^7$

$$\begin{aligned}\cos^5\theta\sin^7\theta &= \frac{1}{2^5}(x + \frac{1}{x})^5\frac{1}{(2i)^7}(x - \frac{1}{x})^7 \\ &= \frac{1}{2^{12}i^7}(x + \frac{1}{x})^5(x - \frac{1}{x})^5(x - \frac{1}{x})^2 \\ &= \frac{1}{2^{12}i^7}(x^2 - \frac{1}{x^2})^5(x - \frac{1}{x})^2\end{aligned}$$ ……… (3)

Using Binomial theorem: $(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}$ we write,

$$\cos^5\theta\sin^7\theta = \frac{1}{2^{12}i^7}\Big[(x^2)^5 - \binom{5}{1}(x^2)^4(\frac{1}{x^2}) + \binom{5}{2}(x^2)^3(\frac{1}{x^2})^2 - \binom{5}{3}(x^2)^2(\frac{1}{x^2})^3 + \binom{5}{4}x^2(\frac{1}{x^2})^4 - (\frac{1}{x^2})^5\Big](x-\frac{1}{x})^2$$

We know that

$\binom{5}{1} = 5$

$\binom{5}{2} = \frac{5*4}{2*1} = 10$

$\binom{5}{3} = \frac{5*4*3}{3*2*1} = 10$

$\binom{5}{4} = 10$

$\binom{5}{5} = 1$

Substituting these values in the above equation, we get

$$\begin{aligned}\cos^5\theta\sin^7\theta &= \frac{1}{2^{12}i^7}\Big[x^{10} - 5x^8\frac{1}{x^2} + 10x^6\frac{1}{x^4} - 10x^4\frac{1}{x^6} + 5x^2\frac{1}{x^8} - \frac{1}{x^{10}}\Big](x-\frac{1}{x})^2 \\ &= \frac{1}{2^{12}i^7}\Big[x^{10} - 5x^6 + 10x^2 - 10\frac{1}{x^2} + 5\frac{1}{x^6} - \frac{1}{x^{10}}\Big](x^2 - 2 + \frac{1}{x^2})\end{aligned}$$

$$\cos^5\theta\sin^7\theta = \frac{1}{2^{12}i^7}\Big[x^{12} - 5x^8 + 10x^4 - 10 + 5\frac{1}{x^4} - \frac{1}{x^8} - 2x^{10} + 10x^6 - 20x^2 + 20\frac{1}{x^2} - 10\frac{1}{x^6} + 2\frac{1}{x^{10}} + x^8 - 5x^4 + 10 - 10\frac{1}{x^4} + 5\frac{1}{x^8} - \frac{1}{x^{12}}\Big]$$

$$\cos^5\theta\sin^7\theta = \frac{1}{2^{12}i^7}\Big[x^{12} - 4x^8 + 5x^4 - 5\frac{1}{x^4} + 4\frac{1}{x^8} - 2x^{10} + 10x^6 - 20x^2 + 20\frac{1}{x^2} - 10\frac{1}{x^6} + 2\frac{1}{x^{10}} - \frac{1}{x^{12}}\Big]$$

$$\cos^5\theta\sin^7\theta = \frac{1}{2^{12}i^7}\Big[(x^{12} - \frac{1}{x^{12}}) - 2(x^{10} - \frac{1}{x^{10}}) - 4(x^8 - \frac{1}{x^8}) + 10(x^6 - \frac{1}{x^6}) + 5(x^4 - \frac{1}{x^4}) - 20(x^2 - \frac{1}{x^2})\Big]$$

From (3), we have

$$\cos^5\theta\sin^7\theta = \frac{1}{2^{12}i^7}\Big[2i\sin 12\theta - 2\cdot 2i\sin 10\theta - 4\cdot 2i\sin 8\theta + 10\cdot 2i\sin 6\theta + 5\cdot 2i\sin 4\theta - 20\cdot 2i\sin 2\theta\Big]$$
$$= \frac{1}{2^{11}i^6}\Big[\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta\Big]$$

We know that $i^6 = i^4\cdot i^2 = -1$, hence

$$\cos^5\theta\sin^7\theta = \frac{-1}{2^{11}}\Big[\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta\Big]$$

Now, we find the nth derivative:

$$D^n[\cos^5\theta\sin^7\theta] = \frac{-1}{2^{11}}D^n\Big[\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta\Big]$$

Using the formula$D^n\sin(ax+b) = \mathrm{a}^{\mathrm{n}}\sin(\frac{n\pi}{2} + ax + b)$, we have

$$D^n[\cos^5\theta\sin^7\theta] = \frac{-1}{2^{11}}\Big[12^n\sin(\frac{n\pi}{2}+12\theta) - 2\cdot 10^n\sin(\frac{n\pi}{2}+10\theta) - 4\cdot 8^n\sin(\frac{n\pi}{2}+8\theta)$$
$$+ 10\cdot 6^n\sin(\frac{n\pi}{2}+6\theta) + 5\cdot 4^n\sin(\frac{n\pi}{2}+4\theta) - 20\cdot 2^n\sin(\frac{n\pi}{2}+2\theta)\Big]$$

**Exercise 2:**

1.Find the $n^{th}$differential coefficient of

(a) $\sin^3 x\cos^5 x$

(b) $sinx\ sin\ 2x\ sin3x$

(c) $e^x sinx$

(d) $\frac{x^2}{(x+1)^2(x+2)}$

## 1.4. Formation of equations involving derivatives

When a relation between x and y is given, we can in many cases deduce from it a relation between the variables x, y and the derivatives of y with respect to x

**Example 1:** If $xy = ae^x + be^{-x}$, prove that $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0$

**Solution:** Given $xy = ae^x + be^{-x}$

Differentiate with respect to x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = ae^x + be^{-x}$$

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = ae^x + be^{-x}$$

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy$$

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0$$

**Example 2:** Prove that if $y = \sin(m\sin^{-1}x)$

**Solution:** Given $y = \sin(m\sin^{-1}x)$

$\sin^{-1}y = m\sin^{-1}x$

Differentiate with respect to x, we get $\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = m\frac{1}{\sqrt{1-x^2}}$

Squaring on both sides, we get

$$\frac{1}{1-y^2}\left(\frac{dy}{dx}\right)^2 = m^2\frac{1}{1-x^2}$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2(1-y^2)$$

Differentiate again with respect to x, we get

$$(1-x^2)2\frac{dy}{dx}\frac{d^2y}{dx^2} + (-2x)\left(\frac{dy}{dx}\right)^2 = m^2(-2y)\frac{dy}{dx}$$

$$2\frac{dy}{dx}\left[(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx}\right] = -2m^2y\frac{dy}{dx}$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -m^2y$$

$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$

$(1-x^2)\, y_2 - xy_1 + m^2y = 0$

**Example 3:**

If $x = \sin\theta$ , $y = \cos p\theta$, prove that $(1-x^2)y_2 - xy_1 + p^2y = 0$

**Solution:**

Given $x = \sin\theta$ , $y = \cos p\theta$

Differentiate both x and y with respect to $\theta$,

$\frac{dx}{d\theta} = \cos\theta, \frac{dy}{d\theta} = -p\sin p\theta$

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-p\sin p\theta}{\cos\theta}$ -----(1)

We know that, $\sin^2 p\theta + \cos^2 p\theta = 1$

$\sin^2 p\theta = 1 - \cos^2 p\theta$

$\sin^2 p\theta = 1 - y^2$

$\sin p\theta = \sqrt{1-y^2}$ ----(2)

Similarly, we have

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$\cos^2\theta = 1 - x^2$

$\cos\theta = \sqrt{1-x^2}$ ……(3)

Sub (2) & (3) in (1)

We get, $\frac{dy}{dx} = -p\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

Squaring on both sides, we get

$$\left(\frac{dy}{dx}\right)^2 = p^2\frac{1-y^2}{1-x^2}$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = p^2(1-y^2)$$

Differentiate with respect to x,

$$(1-x^2)2\frac{dy}{dx}\frac{d^2y}{dx^2} + (-2x)\left(\frac{dy}{dx}\right)^2 = p^2(-2y)\frac{dy}{dx}$$

$$2\frac{dy}{dx}\left((1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx}\right) = -2p^2y\frac{dy}{dx}$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -p^2y$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$$

$$(1-x^2)y_2 - xy_1 + p^2y = 0$$

**Exercise 3:**

1. If $y = ax\ cosmx,$ prove that $x^2\left(\frac{d^2y}{dx^2} + m^2y\right) = 2\left(x\frac{dy}{dx} - y\right)$
2. If x = sin$t$ , y = sin p$t$, prove that $(1-x^2)y_2 - xy_1 + p^2y = 0$
3. If $y = e^{-x}\cos x$ prove that $\frac{d^4y}{dx^4} + 4y = 0$
4. If $y = Ae^{-kt}\cos(pt + e)$, show that $\frac{d^2y}{dx^2} + 2k\frac{dy}{dx} + n^2y = 0$, where $n^2 = p^2 + k^2$

**1.5. Leibnitz formula for the $n^{th}$ derivative of a product:**

If $u$ and $v$ are functions of $x$, we have $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$

This formula express the $n^{th}$ derivative of the product of two variables in terms of the variables themselves and the successive derivatives.

If $u$ and $v$ are functions of $x$, we have $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$

$D(uv) = uDu + uDv$

Differentiating again with respect to $x$

$D^2(uv) = D(vDu) + D(uDv)$

$= vD^2u + uD^2v$

Similarly, $D^3(uv) = vD^3u + 3D^2uDv + 3.DuD^2v + uD^3v$

However, this process will may be continued it will be seen the numeral coefficient follow the same law as that of the binomial theorem and indices of the derivative correspond to the exponents of the binomial theorem.

Hence $n^{th}$ derivative;

$$\frac{d^n}{dx^n}(uv) = \frac{d^n u}{dx^n}v + n_{c_1}\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx} + n_{c_2}\frac{d^{n-2}u}{dx^{n-2}}\frac{d^2v}{dx^2} + \cdots + n_{c_r}\frac{d^{n-r}u}{dx^{n-r}}\frac{d^rv}{dx^r} + \cdots + n_{c_1}\frac{du}{dx}\frac{d^{n-1}u}{dx^{n-1}} + u\frac{d^nv}{dx^n}$$

**Example 1:**

Find the $n^{th}$ differential coefficient of $x^2logx$

**Solution:**

Taking $v = x^2$ and $u = logx$

$$\frac{d^n}{dx^n}(x^2logx) = \frac{d^n}{dx^n}(logx)x^2 + n_{c_1}\frac{d^{n-1}}{dx^{n-1}}(logx)\frac{d}{dx}x^2 + n_{c_2}\frac{d^{n-2}}{dx^{n-2}}(logx)\frac{d^2}{dx^2}x^2$$

All the other terms will be zero since the successive derivatives of $x^2$ after the second derivatives vanish.

$$\therefore D^n(x^2logx) = \frac{(-1)^{n-1}(n-1)!}{x^n}x^2 + \frac{n(-1)^{n-2}(n-2)!}{x^{n-1}}2x + \frac{n(n-1)(-1)^{n-3}(n-3)!}{2x^{n-2}}$$

$$= (-1)^{n-1}(n-1)!\,x^{-n}x^2 + n(-1)^{n-2}(n-2)!\,x^{1-n}2x + \frac{n(n-1)(-1)^{n-3}(n-3)!}{2}x^{2-n}2$$

$$= (n-1)(n-2) - 2n(n-2) + n(n-1)[\,(-1)^{n-3}(n-3)!\ x^{2-n}]$$

$$= n^2 - 2n - n + 2 - 2n^2 + 4n + n^2 - n[(-1)^{n-3}(n-3)!\ x^{2-n}]$$

$$= 2(-1)^{n-3}(n-3)!\ x^{2-n}$$

$$= \frac{2(-1)^{n-3}(n-3)!}{x^{n-2}}$$

**Example 2:**

If $y = \sin(m \sin^{-1} x)$ Prove that $(1-x^2)y_2 - xy_1 + m^2 y = 0$ and $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$

**Solution:**

Given $y = \sin (m\sin^{-1}x)$

$\text{Sin}^{-1}y = m\sin^{-1}x$

Differentiate with respect to x, we get

$$\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = m\,\frac{1}{\sqrt{1-x^2}}$$

Squaring on both sides, we get

$$\frac{1}{1-y^2}\left(\frac{dy}{dx}\right)^2 = m^2\,\frac{1}{1-x^2}$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2\,(1-y^2)$$

Differentiate again with respect to x, we get

$$(1-x^2)2\,\frac{dy}{dx}\frac{d^2y}{dx^2} + (-2x)\left(\frac{dy}{dx}\right)^2 = m^2\,(-2y)\,\frac{dy}{dx}$$

$$2\,\frac{dy}{dx}\left[(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx}\right] = -2m^2y\frac{dy}{dx}$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -m^2y$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$$

$(1-x^2)\,y_2 - xy_1 + m^2y = 0$ …….. (1)

Using Leibnitz theorem of differentiating each term of (1) n times

$D^n(1-x^2)y_2 = y_{n+2}(1-x^2) + ny_{n+1}(-2x) + \frac{n(n-1)}{1\cdot 2}y_n(-2)$ ……..(A)

$D^n(-xy_1) = -y_{n+1}(x) - ny_1(1)$ ………(B)

$D^n(m^2y) = m^2y_n$ .........(C)

Adding RHS terms of (A), (B), and (C)

$y_{n+2}(1-x^2) + y_{n+1}[-2nx - x] + y_n[-n^2 + n - n + m^2] = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$

**Exercise 4:**

1. Find the $n^{th}$ differential coefficient of

(a) $x^2e^{3x}$

(b) $x\ sinx$

(c) $x^2\ cosx$

2. If $y = \sin^{-1}x$ Prove that $(1-x^2)y_2 - xy_1 = 0$ and $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$
3. If $y = \frac{\log x}{x^2}$ show that$x^3\frac{d^3y}{dx^3} + 8x^2\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 4y = 0$

## Unit-2 PARTIAL DIFFERENTIATION:

Partial derivatives – Successive partial derivatives –Function of a function rule – Total differential coefficient.

### PARTIAL DIFFERENTIATION

### 2.1. Partial derivatives:

We have considered till now only functions of one variable but we come across function involving more than one variable. For example, the area of a rectangle is a function of two variables, the length and breadth of the rectangle.

If u be a function of two independent variables x and y, let us assume the functional relation as $u = f(x, y)$. Here x alone or y alone or both x and y are independent, x may be supposed to vary when y remains constant or the reverse.

The derivative of u with respect to x when x varies and y remains constant is called the partial derivative of u with respect to x and is denoted by the symbol $\frac{\partial u}{\partial x}$. We may then write

$$\frac{\partial u}{\partial x} = \underset{\Delta x \to 0}{\text{Lt}} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

Similarly, when x remains constant and y varies, the partial derivative of u with respect to y is

$$\frac{\partial u}{\partial y} = \underset{\Delta y \to 0}{\text{Lt}} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$\frac{\partial u}{\partial x}$ is also written as $\frac{\partial}{\partial x} f(x, y)$ or $\frac{\partial f}{\partial x}$

Similarly, $\frac{\partial u}{\partial y}$ is also written as $\frac{\partial}{\partial y} f(x, y)$ or $\frac{\partial f}{\partial y}$

### 2.2. Successive partial derivatives:

Consider the function $u = f(x, y)$. Then in general $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are functions both x and y and may be differentiated again with respect to either of the independent variables giving rise to successive partiable derivatives. Regarding x alone as varying we denote the result by $\frac{\partial^2 u}{\partial x^2}$ , $\frac{\partial^3 u}{\partial x^3}$ ,…….. , $\frac{\partial^n u}{\partial x^n}$ or when y alone varies , $\frac{\partial^2 u}{\partial y^2}$ , $\frac{\partial^3 u}{\partial y^3}$ ,…….. , $\frac{\partial^n u}{\partial y^n}$

If we differentiate u with respect to x regarding y constant and then this result is differentiated with respect to y regarding x as constant, we obtain $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$ which we denoted by $\frac{\partial^2 u}{\partial y\,\partial x}$.

Similarly, if we differentiate u twice with respect to x and then once with respect to y, the result is denoted by the symbol $\frac{\partial^3 u}{\partial y\,\partial^2 x}$ . The partial differential coefficient of $\frac{\partial u}{\partial y}$ with respect to x considering y as a constant is denoted by $\frac{\partial^2 u}{\partial x\,\partial y}$.

Generally, in the ordinary functions which we come across

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}$$

**2.3. Function of function rule:**

This rule is very useful in partial differentiation.

Let z be a function of u where u is a function of two independent variables x and y.

Then $\frac{\partial z}{\partial x} = \frac{dz}{du}\ \frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y} = \frac{dz}{du}\ \frac{\partial u}{\partial y}$

Let x and y receive arbitrary increments $\Delta x$ and $\Delta y$ and let the corresponding increments in u and z be $\Delta u$ and $\Delta z$ respectively.

Then $\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta u}\ \frac{\Delta u}{\Delta x}$

Proceeding to the limit when $\Delta x \to 0$, $\frac{\partial z}{\partial x} = \frac{dz}{du}\ \frac{\partial u}{\partial x}$

**Note:**

The straight limit d is used in $\frac{dz}{dx}$ as z is a function of only one variable u while the curved $\partial$ is used in $\frac{\partial u}{\partial x}$ as u is a function of two independent variables.

**Example 1:**

Find the partial differential coefficient of u=sin(ax+by+cz)

**Solution:**

Let u=sin(ax+by+cz)

$\frac{\partial u}{\partial x} = a\cos(ax+by+cz)$

$\frac{\partial u}{\partial y} = b\cos(ax+by+c)$

$\frac{\partial u}{\partial z} = c\cos(ax+by+c)$

**Example 2:**

If $u = \frac{xy}{x+y}$. Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$

**Solution:**

$\frac{\partial u}{\partial x} = \frac{(x+y)y-xy}{(x+y)^2}$

$= \frac{y^2}{(x+y)^2}$

Similarly, $\frac{\partial u}{\partial y} = \frac{x^2}{(x+y)^2}$

$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x^2y+xy^2}{(x+y)^2}$

$= \frac{xy}{x+y}$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$

**Example 3:**

If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

**Solution:**

$u = \tan^{-1} \frac{x^3+y^3}{x-y}$

$\tan u = \frac{x^3+y^3}{x-y}$

Differentiate with respect to x,

$$\sec^2\frac{\partial u}{\partial x} = \frac{(x-y)3x^2-(x^3+y^3)}{(x-y)^2}$$

$$= \frac{3x^3-3x^2y-x^3-y^3}{(x-y)^2}$$

$$= \frac{2x^3-3x^2y-y^3}{(x-y)^2}$$

Differentiate with respect to y,

$$\sec^2\frac{\partial u}{\partial y} = \frac{(x-y)3y^2-(x^3+y^3)-1}{(x-y)^2}$$

$$= \frac{3xy^2-3y^3+x^3+y^3}{(x-y)^2}$$

$$= \frac{3xy^2-2y^3+x^3}{(x-y)^2}$$

$$\sec^2 u(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}) = \frac{x(2x^3-3x^2y-y^3)}{(x-y)^2} + \frac{y(3xy^2-2y^3+x^3)}{(x-y)^2}$$

$$= \frac{x(2x^3-3x^2y-y^3)+y(3xy^2-2y^3+x^3)}{(x-y)^2}$$

$$= \frac{2x^4-3x^3y-xy^3+3xy^3-2y^4+x^3y}{(x-y)^2}$$

$$= \frac{2x^4-2x^3y+2xy^3-2y^4}{(x-y)^2}$$

$$= \frac{2(x^4-x^3y+xy^3-y^4)}{(x-y)^2}$$

$= \frac{2(x-y)(x^3+y^3)}{(x-y)^2}$

$= \frac{2(x^3+y^3)}{x-y}$

We know that $\frac{x^3+y^3}{x-y}$ = tan u

= 2tan u

$= \frac{1}{sec^2}$ 2tan u

= 2 $\cos^2$u.tan u

= 2 $\cos^2 u.\frac{\sin u}{\cos u}$

= 2cosu.sinu = sin2u

**Example 4:**

If $v = (x^2+y^2+z^2)^{-1/2}$, show that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$

**Solution:**

Differentiate with respect to x,

$\frac{\partial v}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}.2x$

$= -x\,(x^2+y^2+z^2)^{-3/2}$

Again differentiate with respect to x,

$\frac{\partial^2 v}{\partial x^2} = \frac{3}{2}(x^2+y^2+z^2)^{-5/2}. - (x^2+y^2+z^2)^{-3/2}$

$= \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$

Similarly, $\frac{\partial^2 v}{\partial y^2} = \frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$

$\frac{\partial^2 v}{\partial z^2} = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=0$$

**Example 5:**

Illustrate the theorem that $\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial^2 u}{\partial y\,\partial x}$ when u is equal to log $\frac{x^2+y^2}{xy}$

**Solution:**

$$u=\log\frac{x^2+y^2}{xy}=\log(x^2+y^2)-\log x-\log y$$

$$\frac{\partial u}{\partial x}=\frac{2x}{x^2+y^2}-\frac{1}{x}$$

$$\frac{\partial^2 u}{\partial y\,\partial x}=\frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2}-\frac{1}{x}\right)$$

$$=\frac{-4xy}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y}=\frac{2y}{x^2+y^2}-\frac{1}{y}$$

$$\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2}-\frac{1}{y}\right)$$

$$=\frac{-4xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y\,\partial x}=\frac{\partial^2 u}{\partial x\,\partial y}$$

**Exercise 1:**

1. If $u=\log(\tan x+\tan y+\tan z)$, show that $\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}+\sin 2z\frac{\partial u}{\partial z}=2$
2. If $u=(y-z)(z-x)(x-y)$, show that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$
3. Verify that $\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial^2 u}{\partial y\,\partial x}$ in the following cases:

   (a) $u=sin^{-1}\frac{y}{x}$

   (b) $u=x\,siny+y\,sinx$

   (c) $u=x^y$

   (d) $u=\log\{x\,tan^{-1}(x^2+y^2)\}$

## 2.4. Total differential coefficient

Then $\frac{du}{dx} = f'_x(x,y)\frac{dx}{dt} + f'_y(x,y)\frac{dy}{dt}$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

In the differential form, this can be written as

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$du$ is called the total differential of u.

In the same way, if $u = f(x,y,z)$ and $x, y, z$ are all functions of t, we get

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

And similarly if $u = f(x_1, x_2, \ldots\ldots x_n)$ where $x_1, x_2, \ldots\ldots x_n$ are known functions of a variable t , we have the relation.

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}\frac{dx_2}{dt} + \cdots \ldots . + \frac{\partial u}{\partial x_n}\frac{dx_n}{dt}$$

(or)

$$du = \frac{\partial u}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2 + \cdots \ldots . + \frac{\partial u}{\partial x_n}dx_n$$

**Example 1:**

Find $\frac{du}{dt}$ where $u = x^2 + y^2 + z^2$, $x = e^t$, $y = e^t \sin t$ and $z = e^t \cos t$.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

**Solution:**

$u = x^2 + y^2 + z^2$

$x = e^t$

$y = e^t \sin t$

$z = e^t \cos t$

$$\frac{\partial u}{\partial x} = 2x$$

$$\frac{dx}{dt} = e^t$$

$$\frac{\partial u}{\partial y} = 2y$$

$$\frac{dy}{dt} = e^t \cos t + e^t \sin t$$

$\frac{\partial u}{\partial z} = 2z$ ,$\frac{dz}{dt} = -e^t \sin t + e^t \cos t$

$$\frac{du}{dt} = 2xe^t + 2y(e^t \cos t + e^t \sin t) + 2z(-e^t \sin t + e^t \cos t)$$

$$\frac{du}{dt} = 2e^t[x + y \sin t + y \cos t + z \cos t - z \sin t]$$

$$\frac{du}{dt} = 2e^t[e^t + e^t \sin^2 t + e^t \sin t \cos t + e^t \cos^2 t - e^t \cos t \sin t]$$

$$\frac{du}{dt} = 2e^t[e^t + e^t (\sin^2 t + \cos^2 t)]$$

$$\frac{du}{dt} = 2e^t.2e^t$$

$$\frac{du}{dt} = 4e^t$$

**Example 2:**

Find $\frac{du}{dt}$ , $u = x^3y^4z^2$ where $x = t^2, y = t^3, z = t^4$

**Solution:**

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$\frac{du}{dt} = 3x^2y^4z^2(2t) + 4y^3x^3z^2(3t^2) + 2zx^3y^4(4t^3)$$

$$\frac{du}{dt} = 6(t^4t^{12}t^8t) + 12(t^9t^6t^8t^2) + 8(t^3t^4t^6t^{12})$$

$$\frac{du}{dt} = 6t^{25} + 12t^{25} + 8t^{25}$$

$$\frac{du}{dt} = 26t^{25}$$

**Example 3:**

Find $\frac{du}{dt}$, $u = xyz$ where $x = e^{-t}, y = e^{-t}\sin^2 t, z = \sin t$

**Solution:**

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$\frac{du}{dt} = yz(-e^{-t}) + xz(e^{-t}2\sin\ t\cos t - e^{-t}\sin^2 t) + xy\cos t$$

$$\frac{du}{dt} = e^{-t}\sin^3 t\,(-e^{-t}) + e^{-t}\sin t\,(e^{-t}2\sin\ t\cos t - e^{-t}\sin^2 t) + e^{-2t}\sin^2 t\cos t$$

$$\frac{du}{dt} = e^{-2t}\sin^3 t\left[-1 + \frac{2\cos t}{\sin t} - 1 + \frac{\cos t}{\sin t}\right]$$

$$\frac{du}{dt} = e^{-2t}\sin^3 t\,[3\cos t - 2]$$

**Example 4:**

If $u = \sin(xy^2)$, where $x = \log t, y = e^t$ then prove that $\frac{du}{dt}$.

**Solution:**

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$\frac{du}{dt} = \cos(xy^2)\,(y^2)\left(\frac{1}{t}\right) + \cos(xy^2)(2xy)(e^t)$$

$$\frac{du}{dt} = \cos(xy^2)\left[y^2\left(\frac{1}{t}\right) + 2xy(e^t)\right]$$

$$\frac{du}{dt} = \mathrm{y}^2\cos(xy^2)\left[\frac{1}{t} + 2x(e^t)\right]$$

**Exercise 2:**

1.Find $\frac{du}{dt}$, $u = \log(x + y + z)$ where $x = \cos t, y = \sin^2 t$ and $z = \cos^2 t$

**UNIT- 3 PARTIAL DIFFERENTIATION (Continued)**:

Homogeneous functions – Partial derivativesof a function of two variables - Lagrange's method of undetermined multipliers.

## PARTIAL DIFFERENTIATION

### 3.1. HOMOGENOUS FUNCTIONS

Let us consider the function

$$f(x,y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n$$

In this expression the sum of the indices of the variables x and y in each term is n. Such an expression is called a homogeneous function of degree n. This expression can be written as follows,

$$f(x,y) = x^n(a_0 + a_1\frac{y}{x} + a_2\frac{y^2}{x^2} + \cdots a_n\frac{y^n}{x^n})$$

$$= x^n(\, a\ function\ of\ \tfrac{y}{x})$$

$$= x^n\, F\left(\frac{y}{x}\right)$$

Similarly, a homogenous function of degree n consisting of m variables $x_1, x_2 \ldots \ldots . x_m$ can be written as $x_r{}^n\, F\left(\frac{x_1}{x_r}, \frac{x_2}{x_r} \ldots .. \frac{x_m}{x_r}\right)$

**Theorem 1:**

**Euler's Theorem:** If $f(x,y)$ is a homogeneous function of degree n, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$.

This is known as Euler's Theorem on homogenous functions.

**Proof:**

$$f(x,y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n$$

$$= x^n\, F\left(\frac{y}{x}\right)$$

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[x^n\, F\left(\frac{y}{x}\right)\right]$$

$= nx^{n-1}\, F\left(\frac{y}{x}\right)$-$x^n\, F\left(\frac{y}{x}\right) \cdot \frac{y}{x^2}$

$\frac{\partial f}{\partial x} = nx^{n-1}\, F\left(\frac{y}{x}\right)$-$x^{n-2}\, yF\left(\frac{y}{x}\right)$ ……….(1)

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[x^n\, F\left(\frac{y}{x}\right)\right]$$

$= x^n\, F\left(\frac{y}{x}\right) \cdot \frac{1}{x}$

$\frac{\partial f}{\partial x} = x^{n-1}\, F\left(\frac{y}{x}\right)$ …… (2)

From the equation (1) and (2)

$\therefore x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = x[nx^{n-1}\, F\left(\frac{y}{x}\right)$-$x^{n-2}\, yF\left(\frac{y}{x}\right)]$+$y\left[\, x^{n-1}\, F\left(\frac{y}{x}\right)\right]$

$= nx^n\, F\left(\frac{y}{x}\right)$ - $x^{n-1}\, y\, F\left(\frac{y}{x}\right) + x^{n-1}\, y\, F\left(\frac{y}{x}\right)$

$= nx^n\, F\left(\frac{y}{x}\right)$

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

In general if $f(x_1, x_2 \dots \dots . x_m)$ is a homogeneous function of degree n, then

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \cdots \dots + x_m\frac{\partial f}{\partial x_m} = nf$$

**Example 1:**

Verify Euler's theorem when $u = x^3 + y^3 + z^3 + 3xyz$.

**Solution:**

$$\frac{\partial u}{\partial x} = 3x^2 + 3yz$$

$$\frac{\partial u}{\partial y} = 3y^2 + 3zx$$

$$\frac{\partial u}{\partial z} = 3z^2 + 3xy$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = x(3x^2 + 3yz) + y(3y^2 + 3zx) + z(3z^2 + 3xy)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3(x^3 + y^3 + z^3 + 3xyz)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u$$

**Example 2:**

If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = \sin 2u$.

**Solution:**

$\tan u = \frac{x^3+y^3}{x-y} = x^2\frac{1+\left(\frac{y}{x}\right)^3}{1-\left(\frac{y}{x}\right)} = x^2 f\left(\frac{y}{x}\right)$, which is a homogenous function of degree 2.

Let v = tan u.

Then v is a homogenous function of x and y of degree 2.

$$\therefore x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2v$$

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial x}(\tan u) = 2\tan u$$

$$x\sec^2 u\,\frac{\partial u}{\partial x} + y\sec^2 u\,\frac{\partial u}{\partial y} = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u} = \sin 2u$$

**Exercise 1:**

1.Verify Euler's theorem

(a) $u = x^3 - 3x^2y + 3xy^2 + y^3$.

(b) $u = \sin\left(\frac{x-y}{x+y}\right)^{1/2}$

2. If $u = xy^2 f\left(\frac{y}{x}\right)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u$

3. If $u = \tan^{-1}\frac{x^2+y^2}{x+y}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = \frac{1}{2}\sin 2u$.

## 3.2. PARTIAL DERIVATIVES OF A FUNCTION OF TWO FUNCTIONS

Let $V = F(u, v)$ where $u = f(x, y)$, $v = f_1(x, y)$ and $x, y$ are independent variables.

If we write V in the form $F\{ f(x, y), f_1(x, y)\}$ we can obtain $\frac{\partial V}{\partial x}$ , $\frac{\partial V}{\partial y}$ by the ordinary-rules of partial differentiation but is usually done without substitution.

By definition since $x, y$ are independent

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy \qquad \text{......... (1)}$$

u is a function of $x \; and \; y$

$$\therefore du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad \text{.......... (2)}$$

v is a function of $x \; and \; y$

$$\therefore dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \qquad \text{.......... (3)}$$

V is a function of $u \; and \; v$

$$\therefore dV = \frac{\partial V}{\partial u}du + \frac{\partial V}{\partial v}dv \qquad \text{.......... (4)}$$

Substituting the values of du and dv from (2) and (3) in (4)

We get,

$$dV = \frac{\partial V}{\partial u}\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + \frac{\partial V}{\partial v}\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)$$

$$= \left(\frac{\partial V}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial V}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial y}\right)dy \quad \text{....... (5)}$$

Comparing (1) and (5) , we get

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial y}$$

These results may be expressed by saying that the operators

$\frac{\partial}{\partial x}$ and $\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u}+\frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)$ are equivalent.

Similarly $\frac{\partial}{\partial y}=\frac{\partial u}{\partial y}\frac{\partial}{\partial u}+\frac{\partial v}{\partial y}\frac{\partial}{\partial v}$

$$\frac{\partial^2 V}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)=\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u}+\frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial V}{\partial x}\right)$$

$$\frac{\partial^2 V}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right)=\left(\frac{\partial u}{\partial y}\frac{\partial}{\partial u}+\frac{\partial v}{\partial y}\frac{\partial}{\partial v}\right)\left(\frac{\partial V}{\partial y}\right)$$

In this way, it is possible to express higher partial derivatives.

**Example 1:**

If $z=f(x,y)$ and $x=r\cos\theta$, $y=r\sin\theta$, prove that $\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

**Solution:**

Given that $x=r\cos\theta$

$$\therefore \frac{\partial x}{\partial r}=\cos\theta\,;\ \frac{\partial x}{\partial \theta}=-r\sin\theta$$

$y=r\sin\theta$

$$\therefore \frac{\partial x}{\partial r}=\sin\theta\,;\ \frac{\partial x}{\partial \theta}=r\cos\theta$$

Hence $\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$

$$\frac{\partial z}{\partial r}=\cos\theta\frac{\partial z}{\partial x}+\sin\theta\frac{\partial z}{\partial y}$$

And $\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$

$$\frac{\partial z}{\partial \theta}=-r\sin\theta\frac{\partial z}{\partial x}+r\cos\theta\frac{\partial z}{\partial y}$$

Now, RHS= $\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

$$= \left[(\cos\theta \frac{\partial z}{\partial x} + \sin\theta \frac{\partial z}{\partial y}\right]^2 + \frac{1}{r^2}\left[-r\sin\theta \frac{\partial z}{\partial x} + r\cos\theta \frac{\partial z}{\partial y}\right]^2$$

$$= cos^2\theta \left(\frac{\partial z}{\partial x}\right)^2 + sin^2\theta \left(\frac{\partial z}{\partial y}\right)^2 + 2\cos\theta \frac{\partial z}{\partial x} \sin\theta \frac{\partial z}{\partial y} + \frac{1}{r^2}[r^2 cos^2\theta \left(\frac{\partial z}{\partial y}\right)^2$$

$$+ r^2 sin^2\theta \left(\frac{\partial z}{\partial x}\right)^2 - 2\,\mathrm{r}\cos\theta \frac{\partial z}{\partial x} \sin\theta \frac{\partial z}{\partial y}]$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 [cos^2\theta + sin^2\theta] + \left(\frac{\partial z}{\partial y}\right)^2 [cos^2\theta + sin^2\theta] + 2\cos\theta \frac{\partial z}{\partial x} \sin\theta \frac{\partial z}{\partial y} -$$

$$2\cos\theta \frac{\partial z}{\partial x} \sin\theta \frac{\partial z}{\partial y}$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

=LHS

⇒ RHS=LHS

$$\therefore \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

**Example 2:**

Transform $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}$ into polar coordinates

**Solution:**

$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2 + y^2, \ \tan\theta = \frac{y}{x}$

Differentiating with respect to 'x' we get

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$$

$$\sec^2\theta \frac{\partial\theta}{\partial x} = -\frac{y}{x^2}$$

$$\therefore \frac{\partial\theta}{\partial x} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial V}{\partial\theta}\frac{\partial\theta}{\partial x} = \cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\left(\frac{\partial V}{\partial\theta}\right)$$

Thus $\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$

$$\therefore \frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)$$

$$\frac{\partial^2 V}{\partial x^2} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial\theta}\right)$$

$$\frac{\partial^2 V}{\partial x^2} = \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial\theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial\theta}\right)$$

$$\frac{\partial^2 V}{\partial x^2} = \cos\theta\left[\cos\theta\frac{\partial^2 V}{\partial r^2} - \frac{\sin\theta}{r}\frac{\partial^2 V}{\partial r\partial\theta} + \frac{\sin\theta}{r^2}\frac{\partial V}{\partial\theta}\right] - \frac{\sin\theta}{r}[-\sin\theta\frac{\partial V}{\partial r} + \cos\theta\frac{\partial^2 V}{\partial\theta\partial r} - \frac{\cos\theta}{r}\frac{\partial V}{\partial\theta} - \frac{\sin\theta}{r}\frac{\partial^2 V}{\partial\theta^2}$$

$$\frac{\partial^2 V}{\partial x^2} = \cos^2\theta\frac{\partial^2 V}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 V}{\partial\theta^2} + \frac{\sin^2\theta}{r}\frac{\partial V}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial\theta}$$

Assuming that $\frac{\partial^2 V}{\partial r\partial\theta} = \frac{\partial^2 V}{\partial\theta\partial r}$

To get $\frac{\partial}{\partial y}$, we note that we change $\theta$ in $\frac{\pi}{2} - \theta$

Hence $\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$

Similarly, $\frac{\partial^2 V}{\partial y^2}$ can be found from $\frac{\partial^2 V}{\partial x^2}$ by replacing $\theta$ by $\frac{\pi}{2} - \theta$

This gives

$$\frac{\partial^2 V}{\partial y^2} = \sin^2\theta\frac{\partial^2 V}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 V}{\partial\theta^2} + \frac{\cos^2\theta}{r}\frac{\partial V}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial\theta}$$

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=[\cos^2\theta\frac{\partial^2 V}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\partial\theta}+\frac{\sin^2\theta}{r^2}\frac{\partial^2 V}{\partial\theta^2}+\frac{\sin^2\theta}{r}\frac{\partial V}{\partial r}$$

$$+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial\theta}]+\sin^2\theta\frac{\partial^2 V}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\partial\theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 V}{\partial\theta^2}$$

$$+\frac{\cos^2\theta}{r}\frac{\partial V}{\partial r}+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial\theta}$$

$$\therefore\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=\frac{\partial^2 V}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2}+\frac{1}{r}\frac{\partial V}{\partial r}$$

### 3.3. Lagrange's method of undetermined multipliers:

The function $u=f(x,y,z)+\lambda\,\varphi(x,y,z)$ where $\lambda$ is an undetermined constant. Consider $x,y,z$ as independent variables and write down the conditions $\frac{\partial u}{\partial x}=0$ , $\frac{\partial u}{\partial y}=0$ , $\frac{\partial u}{\partial z}=0$. Solve these three equations along the equation $\varphi(x,y,z)=0$ to find the values of the four quantities $x,y,z$ and $\lambda$.

**Example 1:**

If $u=a^3x^2+b^3y^2+c^3z^2$ where $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$, find the minimum value of u.

**Solution:**

$$u=a^3x^2+b^3y^2+c^3z^2-\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$$

$$\therefore\frac{\partial u}{\partial x}=2a^3x+\frac{\lambda}{x^2}$$

$$\frac{\partial u}{\partial y}=2b^3y+\frac{\lambda}{y^2}$$

$$\frac{\partial u}{\partial z}=2c^3z+\frac{\lambda}{z^2}$$

Equating the expressions

$$2a^3x+\frac{\lambda}{x^2}=0\quad\ldots\ldots\ldots.(1)$$

$$2b^3y + \frac{\lambda}{y^2} = 0 \quad \ldots \ldots \ldots . . (2)$$

$$2c^3z + \frac{\lambda}{z^2} = 0 \quad \ldots \ldots \ldots . . (3)$$

We have $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \quad \ldots \ldots \ldots (4)$

Multiplying equation (1) by x, (2) by y and (3) by z and adding we get

$$2(a^3x^2 + b^3y^2 + c^3z^2) + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$$

(i.e.) $2u + \lambda = 0$

(i.e.) $\lambda = -2u$

Substituting this value of λ in (1), (2) and (3), we get

$$2a^3x - 2\frac{u}{x^2} = 0$$

$$2b^3y - 2\frac{u}{y^2} = 0$$

$$2c^3z - 2\frac{u}{z^2} = 0$$

$$a^3x^3 = b^3y^3 = c^3z^3 = u$$

$$\therefore ax = by = cz = K(say)$$

$$\therefore x = \frac{K}{a}, y = \frac{K}{b}, z = \frac{K}{c}$$

Substituting these values of x, y, z in (4), we have

$$\frac{a}{K} + \frac{b}{K} + \frac{c}{K} = 1$$

(i.e.) $K = a + b + c$

Hence $x = \frac{a+b+c}{a}, y = \frac{a+b+c}{b}, z = \frac{a+b+c}{c}$

Obviously these values will give only the minimum value for the maximum value of u can be obtained by putting x = 1, y = ∞, z = ∞.

The minimum value of u is

$$= \frac{a^3(a+b+c)^2}{a^2} + \frac{b^3(a+b+c)^2}{b^2} + \frac{c^3(a+b+c)^2}{c^2}$$

$$= (a+b+c)^2(a+b+c)$$

$$\Rightarrow (a+b+c)^3$$

**Example 2:**

A tent having the form of a cylinder surmounted by a cone is to contain a given volume. If the canvass required is minimum, show that the altitude of the cone is twice that of the cylinder.

**Solution:**

Let the radius of the cylinder be x, the height of the cylinder be y and the height of the cone be z.

$$volume\ of\ the\ tent = \pi x^2 y + \frac{1}{3}\pi x^2 z$$

$$surface\ of\ the\ tent = 2\pi xy + \pi x\sqrt{x^2 + y^2}$$

The volume of the tent is given as constant and let that be $\pi K^3$.

$$\therefore \pi x^2 y + \frac{1}{3}\pi x^2 z = \pi K^3$$

$$i.e., 3x^2 y + x^2 z = 3K^3 \ldots\ldots\ldots.(1)$$

$$S = \pi(2xy + x\sqrt{x^2 + y^2})$$

S is at a minimum if f (x, y, z) is at a minimum where

$$f(x,y) = 2xy + x\sqrt{x^2 + z^2}$$

So we have to find when $f(x,y) = 2xy + x\sqrt{x^2 + z^2}$ is at a minimum subject to the condition

$$3x^2 y + x^2 z = 3K^3.$$

Let $u = 2xy + x\sqrt{x^2 + z^2} + \lambda(3x^2 y + x^2 z - 3K^3)$

$$\therefore \frac{\partial u}{\partial x} = 2y + \sqrt{x^2 + y^2} + \frac{x^2}{\sqrt{x^2 + y^2}} + 6\lambda xy + 2\lambda xz$$

$$\frac{\partial u}{\partial y} = 2x + 3\lambda x^2$$

$$\frac{\partial u}{\partial z} = \frac{xz}{\sqrt{x^2 + y^2}} + \lambda x^2$$

$$\therefore 2y + \sqrt{x^2 + y^2} + \frac{x^2}{\sqrt{x^2 + y^2}} + 2\lambda(3xy + xz) = 0 \quad \ldots\ldots\ldots.(2)$$

$$2x + 3\lambda x^2 = 0 \quad \ldots\ldots\ldots..(3)$$

$$\frac{xz}{\sqrt{x^2 + y^2}} + \lambda x^2 = 0 \quad \ldots\ldots\ldots\ldots(4)$$

From (3), x = 0 or $2 + 3\lambda x = 0$, *i.e.*, $\lambda x = -\frac{2}{3}$ ……………(5)

x cannot be equal to zero. Hence that value is discarded.

From (4), x = 0 or $\frac{z}{\sqrt{x^2+y^2}} + \lambda x = 0$

From equation (5), substituting the value of λx in this equation, we get

$$\frac{z}{\sqrt{x^2 + y^2}} - \frac{2}{3} = 0$$

$$9z^2 = 4(x^2 + z^2)$$

$$\therefore z^2 = \frac{4}{5}x^2$$

$$z = \pm\frac{2}{\sqrt{5}}x$$

Discarding the negative value, we get $z = \frac{2}{\sqrt{5}}x$ ……………(6)

$$x^2 + z^2 = x^2 + \frac{4}{5}x^2 = \frac{9}{5}x^2$$

Substituting the values of $x^2 + z^2$, λx and z in equation (2), We get $2y + \frac{3}{\sqrt{5}}x + \frac{\sqrt{5}x^2}{3x} + 6\left(-\frac{2}{3}\right)y + 2\left(-\frac{2}{3}\right)\left(\frac{2x}{\sqrt{5}}\right) = 0$

$$2y + \frac{3}{\sqrt{5}}x + \frac{\sqrt{5}}{3}x - 4y - \frac{8x}{3\sqrt{5}} = 0$$

$$2y = \left(\frac{3}{\sqrt{5}} + \frac{\sqrt{5}}{3} - \frac{8}{3\sqrt{5}}\right)x$$

$$2y = \frac{6}{3\sqrt{5}}x \quad \Rightarrow \ y = \frac{x}{\sqrt{5}}$$

**Exercise 2:**

1. Find the minimum value of $x^3 + y^2 + z^2$ when

(i) $xy + yz + zx = 3a^2$

(ii) $xyz = a^3$

2. If $x, y, z$ are the length of the perpendiculars dropped from any point P to the three sides of a triangle of constant area K, show that the minimum value of $x^2 + y^2 + z^2$ is $\frac{4K^2}{a^2+b^2+c^2}$.

**Unit -4 ENVELOPES:**

Method of finding the envelope – Another definition of envelope –Envelope of family of curves which are quadratic in the parameter.

## ENVELOPES

The equation $f(x, y, t) = 0$ determines a curve corresponding to each particular of t. The totality of all such curves by gaining different values of t, is said to be a family of curves and the variable t which is different for different curves is said to be the parameter for the family.

**Examples:**

- The equation $x\, cos\theta + y\, sin\theta = a$, where a is constant represents a family of straight lines for different values of $\theta$ touching the circle $x^2 + y^2 = a^2$. Here $\theta$ is the parameter of the family of straight lines.
- The equation $y = mx + \frac{a}{m}$ represents a family of straight lines with the parameter m touching the parabola $y^2 = 4ax$
- The equation $(x - a)^2 + y^2 = r^2$ where r is a constant is a family of circles with parameter a touching the lines $y = \pm r$.
- The curve E which is touched by a family of curves C is called the envelope of the family of curves C.

**4.1. Method of finding the envelope:**

Let the family of curves C be $f(x, y, t) = 0$ and let us assume that a curve E, the envelope of the family exists and that its equation is $F(x, y) = 0$.

Let us also assume that for a particular value of t, say $\alpha$ it touches E at $(\xi, \eta)$

$\therefore f(\xi, \eta, \alpha) = 0$ ………. (i)

and $F(\xi, \eta) = 0$ ………..(ii)

Considering $\xi, \eta, \alpha$ independent variable and taking total differential in (i) ,

we have $\frac{\partial f}{\partial \xi}\, d\xi + \frac{\partial f}{\partial \eta}\, d\eta + \frac{\partial f}{\partial \alpha}\, d\alpha = 0$

$\frac{\partial f}{\partial \xi}\frac{d\xi}{d\alpha} + \frac{\partial f}{\partial \eta}\frac{d\eta}{d\alpha} + \frac{\partial f}{\partial \alpha} = 0$ ……….. (iii)

Taking total differentials in (ii)

$$\frac{\partial F}{\partial \xi}d\xi + \frac{\partial F}{\partial \eta}d\eta = 0$$

$\frac{\partial F}{\partial \xi}\frac{d\xi}{d\alpha} + \frac{\partial F}{\partial \eta}\frac{d\eta}{d\alpha} = 0$ ………. (iv)

Since the curves $f(x,y,\alpha) = 0$ and $F(x,y) = 0$ touch one another at $(\xi,\eta)$ , their gradients at $(\xi,\eta)$ are equal.

For $f(x,y,\alpha) = 0$ , $\frac{dy}{dx}$ at $(\xi,\eta) = -\frac{\frac{\partial f}{\partial \xi}}{\frac{\partial f}{\partial \eta}}$

For $F(x,y) = 0$ $\frac{dy}{dx}$ at $(\xi,\eta) = -\frac{\frac{\partial F}{\partial \xi}}{\frac{\partial F}{\partial \eta}}$

Hence $\frac{\frac{\partial f}{\partial \xi}}{\frac{\partial f}{\partial \eta}} = \frac{\frac{\partial F}{\partial \xi}}{\frac{\partial F}{\partial \eta}}$

But from (iv) $\frac{\frac{\partial F}{\partial \xi}}{\frac{\partial F}{\partial \eta}} = -\frac{\frac{\partial \eta}{\partial \alpha}}{\frac{\partial \xi}{\partial \alpha}}$

$\therefore \frac{\frac{\partial f}{\partial \xi}}{\frac{\partial f}{\partial \eta}} = -\frac{\frac{\partial \eta}{\partial \alpha}}{\frac{\partial \xi}{\partial \alpha}}$

$\therefore \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial \alpha} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial \alpha} = 0$

Comparing (iii) and (v) we get $\frac{\partial f}{\partial \alpha} = 0$ and this equation is satisfied by $(\xi,\eta)$.

Hence $(\xi,\eta)$ satisfies both the equations $f(x,y,\alpha) = 0$ and $\frac{\partial f}{\partial \alpha} = 0$.

Therefore the envelope of the family of curves $f(x,y,t) = 0$ is got by eliminating t between the equations $f(x,y,t) = 0$ and $\frac{\partial}{\partial t}f(x,y,t) = 0$.

## 4.2. Another definition of envelope

The envelope of a family of curves is the locus of the limiting position of the intersecting points of any two curves of the family when one of them tends to coincide with the other which is fixed.

Let a curve of the family of curves $f(x, y, t) = 0$ be $f(x, y, \alpha) = 0$ …….(i)

Let the curve of the family in the neighbourhood of (i) be

$f(x, y, \alpha + \Delta\alpha) = 0$ ……. (ii)

Let these two curves (i) and (ii) intersect at $P_1$ .Then the coordinates of $P_1$ will satisfy the equation.

$$f(x, y, \alpha + \Delta\alpha) - f(x, y, \alpha) = 0$$

And therefore, also the equation

$$\frac{f(x,y,\alpha+\Delta\alpha) - f(x,y,\alpha)}{\Delta\alpha} = 0$$

Taking limit as $\Delta\alpha \to 0$, we see that the point $P_1 \to P$ and the coordinates of $P$ to which $P_1$ tends as$\Delta\alpha \to 0$ satisfy the equation.

$$\lim_{\Delta\alpha \to 0} \frac{f(x,y,\alpha+\Delta\alpha) - f(x,y,\alpha)}{\Delta\alpha} = 0$$

$$\frac{\partial}{\partial\alpha} f(x, y, \alpha) = 0$$

Hence P is a point which satisfies both the equations

$$f(x, y, \alpha) = 0 \ \ and \ \frac{\partial}{\partial\alpha} f(x, y, \alpha) = 0$$

Therefore by eliminating $\alpha$ from the above two equations, we get the locus of P, which is the same as the rule obtained in 4.1.

## 4.3. Envelope of family of curves which are quadratic in the parameter

When $f(x, y, t) = 0$ is merely a quadratic in t, say $At^2 + Bt + C = 0$, where A,B,C are functions of x, y and t is the parameter, the envelope is obtained by eliminating t between the equations,

$At^2 + Bt + C = 0$ ……. (i)

Differentiating with respect to t,

$2At + B = 0$ ……… (ii)

From (ii), $t = \frac{-B}{2A}$ and Substituting this value of t in (i), the equation of the envelope is $B^2 = 4AC$.

**Example 1:**

Find the envelope of the family of a straight lines $y + tx = 2at + at^3$, the parameter being t.

**Solution:**

Given that $y + tx = 2at + at^3$ where t is the parameter

Differentiating partially with respect to t, we have

$x = 2a + 3at^2$ … … … $(i)$

To get the envelope we have to eliminate t from (i) and the equation

$y + tx = 2at + at^3$

From (ii),

$y = t(-x + 2a + at^2)$

Substituting the value of $t^2$ from (i) in this equation, we have

$$y = t\left(-x + 2a + \frac{x - 2a}{3}\right)$$

$$y = -\frac{2t}{3}(x - 2a)$$

Hence $y^2 = \frac{4t^2}{9}(x - 2a)^2$

$$y^2 = \frac{4}{9}\left(\frac{x - 2a}{3a}\right)(x - 2a)^2$$

$27ay^2 = 4(x - 2a)^3$ which is the equation of the required envelope.

This curve is called a semi – cubical parabola.

**Example 2:**

Find the envelope of the family of circles $(x-a)^2+y^2=2a$, where a is the parameter.

**Solution:**

The family of circles is $(x-a)^2+y^2=2a \dots\dots\dots (i)$

Differentiating partially with respect to a, we get

$-2(x-a)=2 \qquad \dots\dots\dots. (ii)$

Eliminate a between (i) and (ii).

Substituting the value of a form (ii) in (i), we have

$(-1)^2+y^2=2(x+1)$

$y^2=2x+1$

**Example 3:**

Find the envelope of the family of curves $\frac{x^2}{a^2}+\frac{y^2}{K^2-a^2}=1$, where a is the parameter.

**Solution:**

Given that $\frac{x^2}{a^2}+\frac{y^2}{K^2-a^2}=1$ , where a is the parameter

$x^2(K^2-a^2)+y^2a^2=a^2(K^2-a^2)$

$a^4-a^2(x^2-y^2+K^2)+x^2K^2=0$

Since this is a equation in $a^2$, its eliminant is

$(x^2-y^2+K^2)^2=4x^2K^2$

$(x^2-y^2+K^2)=\pm 2xK$

$x^2\pm 2xK+K^2=y^2$

$(x\pm K)^2=y^2$

$(x\pm K)=\pm y$

$x\pm y=\pm K$

**Example 4:**

Find the envelope of the circles drawn on the radius vectors of the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as parameter.

**Solution:**

The coordinates of any point P on the ellipse are $(a\cos\theta, b\sin\theta)$.

To find the envelope of this circles midpoint $= \left(\frac{x_1+x_2}{2}\right), \left(\frac{y_1+y_2}{2}\right)$

$$= \left(\frac{a\cos\theta}{2}\right), \left(\frac{b\sin\theta}{2}\right)$$

Diameter $= \sqrt{(a\,cos\theta - 0)^2 + (b\,sin\theta - 0)^2}$

$= \sqrt{a^2cos^2\theta + b^2sin^2\theta}$

Radius $= \frac{\sqrt{a^2cos^2\theta + b^2sin^2\theta}}{2}$

Equation of the circle $\Rightarrow (x - x_0)^2 + (y - y_0)^2 = r^2$

$$\Rightarrow \left(x - \frac{a\cos\theta}{2}\right)^2 + \left(y - \frac{b\sin\theta}{2}\right)^2 = \left(\frac{\sqrt{a^2cos^2\theta + b^2sin^2\theta}}{2}\right)^2$$

$$\Rightarrow 4x^2 + a^2cos^2\theta - 4xa\,cos\theta + 4y^2 + b^2sin^2\theta - -4yb\,sin\theta = a^2cos^2\theta + b^2sin^2\theta$$

$$\Rightarrow 4x^2 + 4y^2 - 4ax\cos\theta - 4by\sin\theta = 0$$

$$x^2 + y^2 - ax\cos\theta - by\sin\theta = 0 \quad \ldots\ldots\ldots(i)$$

We have to find the envelope of the family of circle (i) for different values of $\theta$, we have

$$ax\cos\theta - by\sin\theta = 0 \quad \ldots\ldots\ldots(ii)$$

We have to eliminate $\theta$ from the equations (i) and (ii).

From (ii), $\frac{\cos\theta}{ax} = \frac{\sin\theta}{by}$ which is equal to $\frac{1}{\sqrt{a^2x^2+b^2y^2}}$,

$\therefore \cos\theta = \frac{ax}{\sqrt{a^2x^2+b^2y^2}}$, $\sin\theta = \frac{by}{\sqrt{a^2x^2+b^2y^2}}$

Substituting the values of $\sin\theta$ and $\cos\theta$ in (i), we get

$$x^2 + y^2 - \frac{a^2x^2}{\sqrt{a^2x^2 + b^2y^2}} - \frac{b^2y^2}{\sqrt{a^2x^2 + b^2y^2}} = 0$$

$$x^2 + y^2 = \sqrt{a^2x^2 + b^2y^2}$$

$(x^2 + y^2)^2 = a^2x^2 + b^2y^2$ which is the equation of the required envelope.

**Example 5:**

Find the envelope of the family of curves.

$x^2 + y^2 - 2ax\cos\theta - 2ay\sin\theta = c^2$

**Solution:**

$x^2 + y^2 - c^2 = 2ax\cos\theta + 2ay\sin\theta \dots\dots\dots.(1)$

Differentiating,

$0 = -2ax\sin\theta + 2ay\cos\theta \quad \dots\dots\dots..(2)$

$(1)^2 + (2)^2$,

$$(x^2 + y^2 - c^2)^2 = (2ax\cos\theta + 2ay\sin\theta)^2 + (-2ax\sin\theta + 2ay\cos\theta)^2$$

$$(x^2 + y^2 - c^2)^2$$
$$= 4a^2x^2\cos^2\theta + 4a^2y^2\sin^2\theta + 8a^2xy\cos\theta\sin\theta + 4a^2x^2\sin^2\theta$$
$$+ 4a^2y^2\cos^2\theta - 8a^2xy\cos\theta\sin\theta$$

$(x^2 + y^2 - c^2)^2 = 4a^2x^2(\cos^2\theta + \sin^2\theta) + 4a^2y^2(\cos^2\theta + \sin^2\theta)$

$(x^2 + y^2 - c^2)^2 = 4a^2x^2 + 4a^2y^2$

$(x^2 + y^2 - c^2)^2 = 4a^2(x^2 + y^2)$

It is the required envelope.

**Example 6:**

Find the envelope of the family of curves.

$(x - a)^2 + (y - a)^2 = 4a$

**Solution:**

$(x-a)^2+(y-a)^2=4a \quad \dots\dots\dots\dots(1)$

Differentiating,

$2(x-a)(-1)+2(y-a)(-1)=4$

$-2x+2a-2y+2a=4$

$-2(x-a+y-b)=4$

$x+y-2a=-2$

$-2a=-2-x-y$

$2a=2+x+y$

$a=\frac{2+x+y}{2}$

$(1), \left(x-\frac{2+x+y}{2}\right)^2+\left(y-\frac{2+x+y}{2}\right)^2=4\left(\frac{2+x+y}{2}\right)$

$(2x-(2+x+y))^2+\left(2y-(2+x+y)\right)^2=8(x+y+z)$

$Multiplying\ by\ 2,$

$$\left(4x^2+(x+y+2)^2-4x(x+y+z)\right)+\left(4y^2+(x+y+2)^2-4y(x+y+z)\right)$$
$$=8(x+y+z)$$

$4(x^2+y^2)+2(x+y+2)^2-4(x+y+z)(x+y+z)-8(x+y+z)=0$

$4(x^2+y^2)-2(x+y+z)^2=0 \Rightarrow 2(x^2+y^2)=(x+y+z)^2$

**Example 7:**

Find the envelope of the family of curves
$x^2\cos\theta+y^2\sin\theta=a^2$
**solution**:

$x^2\cos\theta+y^2\sin\theta=a^2 \quad \dots\dots\dots\dots..(1)$

Differentiating,

$$-x^2 \sin\theta + y^2 \cos\theta = 0$$

$$y^2 \cos\theta = x^2 \sin\theta$$

$$\frac{x^2}{y^2} = \frac{\sin\theta}{\cos\theta}$$

$$\sin\theta = \frac{y^2}{\sqrt{x^4+y^4}} \; and \; \cos\theta = \frac{x^2}{\sqrt{x^4+y^4}}$$

Equation (1) implies

$$x^2\left(\frac{x^2}{\sqrt{x^4+y^4}}\right) + y^2\left(\frac{y^2}{\sqrt{x^4+y^4}}\right) = a^2$$

$$\left(\frac{x^4+y^4}{\sqrt{x^4+y^4}}\right) = a^2$$

$$\sqrt{x^4+y^4} = a^2$$

$$x^4 + y^4 = a^4$$

**Example 8:**

Find the envelope of the family of curves $\left(\frac{a^2}{x}\right)\cos\theta - \left(\frac{b^2}{y}\right)\sin\theta = c$

**Solution:**

$$\left(\frac{a^2}{x}\right)\cos\theta - \left(\frac{b^2}{y}\right)\sin\theta = c \quad \dots\dots\dots\dots (1)$$

$$\left(\frac{a^2}{x}\right)(-\sin\theta) - \left(\frac{b^2}{y}\right)\cos\theta = 0$$

$$-\frac{a^2\sin\theta}{x} = \frac{b^2\cos\theta}{y}$$

$$\frac{\sin\theta}{\cos\theta} = -\frac{b^2 x}{a^2 y}$$

$$\sin\theta = -\frac{b^2 x}{\sqrt{a^4y^2+b^4x^2}} \; and \cos\theta = \frac{a^2 y}{\sqrt{a^4y^2+b^4x^2}}$$

$$\left(\frac{a^2}{x}\right)\left(\frac{a^2y}{\sqrt{a^4y^2+b^4x^2}}\right)-\left(\frac{b^2}{y}\right)\left(-\frac{b^2x}{\sqrt{a^4y^2+b^4x^2}}\right)=c$$

$$\left(\frac{a^4y}{x\sqrt{a^4y^2+b^4x^2}}\right)+\left(\frac{b^4x}{y\sqrt{a^4y^2+b^4x^2}}\right)=c$$

$$\frac{a^4y^2+b^4x^2}{xy\sqrt{a^4y^2+b^4x^2}}=c$$

$$\frac{a^4y^2+b^4x^2}{\sqrt{a^4y^2+b^4x^2}}=cxy$$

$$\sqrt{a^4y^2+b^4x^2}=cxy$$

$$a^4y^2+b^4x^2=c^2x^2y^2$$

**Exercise 1:**

1. Find the envelope of the family of a straight lines

(a) $y=t^2x-t^3$ , where t is parameter.

(b) $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$ , where $\theta$ is parameter.

(c) $x\ tan\theta+y\sec\theta=1$ , where $\theta$ is parameter.

**4.4. Envelope of family of curves in two parameters**

Sometimes the family of curves will contain two parameters and the two parameters are connected by a relation. In that case we can express the equation of the family of curves containing only one parameter but the process may be tedious.

**Example 1:**

Find the envelopes of the straight lines $\frac{x}{a}+\frac{y}{b}=1$ where the parameters are related by the equation $a^2+b^2=c^2$ where c is a constant.

**Solution:**

Let us regard a and b as functions of t.

$a^2+b^2=c^2 \qquad \dots\dots\dots(i)$

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \dots \dots \dots \dots (ii)$$

From equation (ii) Differentiating with respect to t, we have

$$-\frac{x}{a^2}\frac{da}{dt} - \frac{y}{b^2}\frac{db}{dt} = 0 \quad \dots \dots \dots \dots (iii)$$

Differentiating $a^2 + b^2 = c^2$ with respect to t, we have

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 0 \quad \dots \dots \dots .. (iv)$$

Comparing (iii) and (iv), we have $\frac{x}{a^3} = \frac{y}{b^3}$ $\quad \dots \dots \dots . (v)$

We have to eliminate a and b from (i), (ii) and (v).

From (v),

$$\frac{\frac{x}{a}}{a^2} = \frac{\frac{y}{b}}{b^2} = \frac{\frac{x}{a} + \frac{y}{b}}{a^2 + b^2} = \frac{1}{c^2}$$

$$\therefore a^3 = c^2 x, b^3 = c^2 y$$

Substituting the values of a and b in (iii), we get,

$$(c^2 x)^{\frac{2}{3}} + (c^2 y)^{\frac{2}{3}} = c^2$$

$x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$ which is the equation of the required envelope (Four cusped hypocyloid).

**UNIT-5 CURVATURE OF PLANE CURVES**

Definition of Curvature – Circle, Radius and Centre of Curvature –Evolutes and Involutes – Radius of Curvature in Polar Co-ordinates.

**CURVATURE OF PLANE CURVES**

**5.1 Definition of Curvature**

Consider a curve given by the equation $y = f(x)$ suppose the curve has a definite tangent at each point.

- Let A be a fixed point on the curve and P be an arbitrary point on the curve
- Let S denote the arc length AP
- Let Y be the angle made by the tangent with the x-axis
- The $\left(\frac{d\psi}{ds}\right)$ is called the curvature of the curve at P

  Thus, the curvature is the rate of turning or bending of the tangent w.r.to the arc length

**5.2 Curvature of a Circle**

**Theorem:1**

Prove that the curvature of a circle of radius r at any point is $\frac{1}{r}$

**Proof:**

Let A be a fixed on the circle and $\phi$ be any point on the circle.

Let arc AP=S and the tangent at P make an angle $\psi$ with tangent at A.

Then, ∠AOP= $\psi$

$s = r\psi$

$$\Rightarrow \frac{ds}{d\psi} = r$$

$$\Rightarrow \frac{d\psi}{ds} = \frac{1}{r}$$

Hence the curvature of a circle of radius $r$ is $\frac{1}{r}$

## 5.3 Radius of Curvature

The reciprocal of curvature of a curve at a point is called the radius of curvature of the curve at the point. So it is $\frac{ds}{d\psi}$.

The radius of Curvature of a circle is its radius.

**Notation**

Radius of Curvature is denoted by ρ.

**Remark :1**

In the case of a straight line the change of Ψ is zero and hence $\frac{d\psi}{ds} = 0$,

$$\rho = \frac{ds}{d\psi} = \infty$$

**Remark: 2**

If the curve is such that, as 's' increases, Ψ increases, then $\frac{d\psi}{ds}$ is positive and, so ρ is positive. (i.e.) if the curve is concave, ρ is positive otherwise is negative.

In general, ρ is given as its absolute value, namely $|\rho|$ .

## 5.4. Radius of curvature of Cartesian curve: $y = f(x)$

We know that $\frac{dy}{dx} = tan\Psi$

$$\frac{d^2y}{dx^2} = sec^2\Psi\frac{d\Psi}{dx} = sec^2\Psi\frac{d\Psi}{ds}\frac{ds}{dx}.$$

$$\frac{ds}{d\psi} = \frac{sec^3\Psi}{\frac{d^2y}{dx^2}} \quad (\because \frac{dx}{ds} = \cos\Psi)$$

$$= \frac{[1 + tan^2\Psi]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\boldsymbol{\rho} = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1+y_1{}^2)^{3/2}}{y_2}$$ **,** When tangent is parallel to x-axis

Where $y_1 = \frac{dy}{dx}$, $y_2 = \frac{d^2y}{dx^2}$

**Example 1:**

What is the radius of curvature of the curve $x^4 + y^4 = 2$ at the point (1,1)

**Solution:**

Given the curve $x^4 + y^4 = 2$

Differentiating with respect to x the above equation, we get

$$4x^3 + 4y^3 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{4x^3}{4y^3}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^3}{y^3}$$

Differentiating this once again with respect to x , we get

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$\frac{d^2y}{dx^2} = \frac{y^3(-3x^2) - (-x^3)3y^2\frac{dy}{dx}}{(y^3)^2}$$

$$= \frac{-3x^2y^3 + 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{(y^3)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-[3x^2y^3 + 3x^3y^2\left(\frac{x^3}{y^3}\right)]}{(y^3)^2}$$

At (1,1) $\Rightarrow \frac{dy}{dx} = -1$ , $\frac{d^2y}{dx^2} = -\left[\frac{3+3}{1}\right] = -6$

Hence the radius of curvature is

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$= \frac{[1+(-1)^2]^{3/2}}{-6} = \frac{(2)^{3/2}}{-6} = \frac{-2\sqrt{2}}{6}$

$\Rightarrow \rho = \frac{-\sqrt{2}}{3}$

**Example 2:**

Show that the radius of curvature at any point of the catenary $y = c\cosh\left(\frac{x}{c}\right)$ is equal to the length of the portion of the normal intercepted between the curve and the axis of x.

**Solution:**

Given that $y = c\cosh\left(\frac{x}{c}\right)$

$\Rightarrow \cosh\left(\frac{x}{c}\right) = \frac{y}{c}$ --------(1)

Differentiating with respect to x

$\frac{dy}{dx} = -c\sinh\left(\frac{x}{c}\right).\frac{1}{c} = \sinh\left(\frac{x}{c}\right)$

Again differentiating with respect to x

$\frac{d^2y}{dx^2} = -\cosh\left(\frac{x}{c}\right).\frac{1}{c}$

Hence the radius of curvature is

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1+sinh^2\left(\frac{x}{c}\right)\right]^{3/2}}{\cosh\left(\frac{x}{c}\right).\frac{1}{c}}$$

$$= \frac{c\left[cosh^2\left(\frac{x}{c}\right)\right]^{3/2}}{cosh\left(\frac{x}{c}\right)}$$

$$= \frac{c\left[cosh^3\left(\frac{x}{c}\right)\right]}{cosh\left(\frac{x}{c}\right)}$$

$$= c\left[cosh^2\left(\frac{x}{c}\right)\right]$$

By equation (1) $\Rightarrow\ \rho = c\frac{y^2}{c^2}$

$$\Rightarrow\ \rho = \frac{y^2}{c}$$

At any point *(x,y)* length of the normal

$$\Rightarrow y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2} = y\ \left[1 + sinh^2\left(\frac{x}{c}\right)\right]^{1/2}$$

$$= y\ \left[cosh^2\left(\frac{x}{c}\right)\right]^{1/2}$$

$$= y\ [cosh\left(\frac{x}{c}\right)]$$

$$= y\left(\frac{y}{c}\right) \quad (\therefore \text{ from equation (1) })$$

$$= \frac{y^2}{c}$$

Radius of curvature= Length of the normal

**Exercise 1:**

1. Find the radius of curvature for the curves

(a) $y = e^x$ at the point where it crosses the y – axis

(b) $\sqrt{x} + \sqrt{y} = 1$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$

(c) $y^2 = x^3 + 8$ at the point (-2, 0).

(d) xy = 30 at the point (3,10)

(e) $(x^2 + y^2)\ 2 = a^2\ (y^2 – x^2)$ at the point (0, a)

## 5.5. Radius of curvature of parametric curve: $\boldsymbol{x = f(\theta)\ y, = g(\theta)}$

$\rho = \frac{\left[x'^2+y'^2\right]^{3/2}}{x'y''-y'x''}$ , where $x' = \frac{dx}{d\theta}$ and $y' = \frac{dy}{d\theta}$

**Example 1:**

If a curve is defined by the parametric equation x=f (θ) and y=ϕ(θ), prove that the curvature is $\frac{1}{\rho} = \frac{x'y''-y'x''}{\left[x'^2+y'^2\right]^{3/2}}$

**Solution:**

where dashes denote differentiation with respect to θ.

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{d\theta}{dx} = \frac{y'}{x'}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y'}{x'}\right) = \frac{d}{d\theta}\left(\frac{y'}{x'}\right)\frac{d\theta}{dx}$$

$$= \frac{y''x'-y'x''}{x'^2}.\frac{1}{x'}$$

$$= \frac{y''x'-y'x''}{x'^3}$$

$$\therefore \frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} = \frac{y''x'-y'x''}{x'^3\left[1+\left(\frac{y'}{x'}\right)^2\right]^{3/2}}$$

$$\Rightarrow \frac{1}{\rho} = \frac{x'y''-y'x''}{\left[x'^2+y'^2\right]^{3/2}}$$

**Example 2:**

Prove that the radius of curvature at any point of the cycloid $x = a(\theta + sin\ \theta)$ and $y = a(1 - cos\ \theta)$ is $4a\cos\frac{\theta}{2}$.

**Solution:**

From the given equations,

$$x = a(\theta + sin\ \theta)$$

Differentiating with respect to $\theta$

$$x' = \frac{dx}{d\boldsymbol{\theta}} = a(1 + cos\theta)$$

$$x'' = \frac{d^2 x}{d\theta^2} = -a\ sin\theta$$

$$y = a(1 - \cos\theta)$$

Differentiating with respect to $\theta$

$$y' = \frac{dy}{d\theta} = asin\ \theta$$

$$y'' = \frac{d^2 y}{d\theta^2} = a\ cos\theta$$

$$\Rightarrow \frac{1}{\rho} = \frac{x'y'' - y'x''}{\left[x'^2 + y'^2\right]^{3/2}}$$

$$\frac{1}{\rho} = \frac{a(1 + cos\theta)a\ cos\theta - asin\ \theta(-a\ sin\theta)}{[(a(1 + cos\theta))^2 + (a\ sin\theta)^2]^{3/2}}$$

$$= \frac{a^2\ (1 +\ cos\theta)}{a^3\ [2(1 +\ cos\theta)]^{3/2}}$$

$$= \frac{2\ cos^2\ \theta/2}{a\ [4\ cos^2\ \theta/2)]^{3/2}}$$

$$= \frac{1}{4a\ cos\theta/2}$$

$$\therefore \rho = 4a \cos\frac{\theta}{2}$$

**Exercise 2:**

1.Prove that the radius of curvature at the point $\theta$ on the curve

$x = 3a\ cos\theta - a\cos 3\theta\ ,\ y = 3a\ sin\theta - a\ sin3\theta$ is $3a\ sin\theta$.

2. Find the radius of curvature at the point $\theta$ on the curve

$x = a \log sec\theta\ , y = a\ (tan\ \theta - \theta)$

## 5.6. The centre of curvature:

$X = x - \frac{y_1(1+{y_1}^2)}{y_2}$

$$Y = y + \frac{(1 + {y_1}^2)}{y_2}$$

The locus of the centre of curvature for a curve is called the evolute of the curve.

**Example 1:**

Find the co- ordinates of the centre of curvature of the curve xy = 2 at the point (2,1).

**Solution:**

Given that $xy = 2$

$$\Rightarrow y = \frac{2}{x}$$

Differentiating with respect to $x$

$$\Rightarrow \frac{dy}{dx} = -\frac{2}{x^2}$$

Again differentiating with respect to $x$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{4}{x^3}$$

At (2,1) the values of $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$ are respectively $\left(-\frac{1}{2}, \frac{1}{2}\right)$

$$X = x - \frac{y_1(1 + {y_1}^2)}{y_2}$$

$$= 2 + \frac{\frac{1}{2}(1+(-\frac{1}{2})^2)}{\frac{1}{2}}$$

$$= 2 + \frac{\frac{1}{2}(1+\frac{1}{4})}{\frac{1}{2}} = \frac{13}{4}$$

$$\Rightarrow \; X = 3\frac{1}{4}$$

$$Y = y + \frac{(1 + {y_1}^2)}{y_2}$$

$$= 1 + \frac{(1 + (-\frac{1}{2})^2)}{\frac{1}{2}}$$

$$= 1 + \frac{(1 + \frac{1}{4})}{\frac{1}{2}} = \frac{7}{2}$$

$$\Rightarrow \; Y = 3\frac{1}{2}$$

The centre of curvature $\left(3\frac{1}{4}, 3\frac{1}{2}\right)$.

**Example 2:**

Show that in the parabola $y^2 = 4ax$ at the point t, $\rho = -2a\,(1 + t^2)^{3/2}$ , $X = 2a + 3at^3$ , $Y = -2at^3$.Deduce the equation of the evolute.

**Solution:**

Let $x = at^2$ , $Y = 2at$

Differentiating with respect to $t$

$\frac{dx}{dt} = 2at$ , $\frac{dy}{dt} = 2a$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{2a}{2at} = \frac{1}{t}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\frac{1}{t} \div \frac{dx}{dt}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{-1/t^2}{2at} = \frac{-1}{2at^3}$$

$$\Rightarrow \rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2 y}{dx^2}}$$

$$= \frac{\left[1 + \left(\frac{1}{t}\right)^2\right]^{3/2}}{\frac{-1}{2at^3}}$$

$$=\left[1 + \left(\frac{1}{t}\right)^2\right]^{3/2} . -2at^3$$

$$\rho = -2a\,(1 + t^2\,)^{3/2}$$

$$\Rightarrow X = x - \frac{y_1(1 + {y_1}^2)}{y_2}$$

$$= at^2 - \frac{\frac{1}{t}(1+(1/t)^2)}{\frac{-1}{2at^3}}$$

$$= at^2 - \frac{(1+t^2)}{t^2}\left(\frac{1}{t}\right). - 2at^3$$

$$X = at^2 + 2at^2 + 2a$$

$$\Rightarrow Y = y + \frac{(1 + {y_1}^2)}{y_2}$$

$$= 2at + \frac{(1+1/t^2)}{\frac{-1}{2at^2}} = -2at^3$$

$$\Rightarrow \quad Y = -2at^3$$

Eliminating t from X and Y,

$$Y = -2a\,\frac{(x - 2a)^{3/2}}{3a}$$

Squaring both sides and simplifying, we get

$$27\,aY^2 = 4(x - 2a)^3$$

The locus of (X,Y) is $27\,aY^2 = 4(x - 2a)^3$.

The curve is called a semi-cubical parabola.

**Example 3:**

Find the evolute of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$.

**Solution:**

Any Point on the ellipse is $(a\,cos\theta\ , b\,sin\theta)$ .

$x = a\,cos\theta\ , y = b\,sin\theta$

Differentiating with respect to $\theta$

$\frac{dx}{d\theta} = -a\,sin\theta\ , \frac{dy}{d\theta} = b\,cos\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{d\theta}{dx} = \frac{b\,cos\theta}{-a\,sin\theta} = -\frac{b}{a}\cot\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx} = \frac{d}{d\theta}\left(-\frac{b}{a}\cot\theta\right)\frac{-1}{a\,sin\theta}$$

$$= \left(-\frac{b}{a^2}(-\text{cosec}^2\theta)\right)\frac{-1}{sin\theta}$$

$$= \left(-\frac{b}{a^2}\frac{1}{sin^2\theta}\right)\frac{-1}{sin\theta}$$

$$\frac{d^2y}{dx^2} = \left(-\frac{b}{a^2 sin^3\theta}\right)$$

$$y_1 = -\frac{b}{a}\cot\theta\ , y_2 = -\frac{b}{a^2 sin^3\theta}$$

Let $(x, y)$ be the centre of curvature

$$\Rightarrow X = x - \frac{y_1(1+{y_1}^2)}{y_2}$$

$$= a\,cos\theta - \frac{-\frac{b}{a}\cot\theta\,(1+\frac{b^2}{a^2}cot^2\theta)}{\frac{b}{a^2}\ cosec^3\theta}$$

$$X = \frac{(a^2-b^2)\,cos^3\theta}{a}$$

$$\Rightarrow Y = y + \frac{(1+{y_1}^2)}{y_2}$$

$$= b\,sin\theta - \frac{(1+\frac{b^2}{a^2}cot^2\theta)}{\frac{b}{a^2}\ cosec^3\theta}$$

$$Y = -\frac{(a^2 - b^2)}{b} \, sin^3 \theta$$

$$= \frac{(a^2 - b^2) \, cos^3 \theta}{a}$$

$$cos\theta = \left(\frac{ax}{a^2-b^2}\right)^{1/3} \;, sin\theta = \left(\frac{-by}{a^2-b^2}\right)^{1/3}$$

To eliminate $\theta$ squaring and adding , we get

$$\left(\frac{ax}{a^2 - b^2}\right)^{\frac{2}{3}} + \left(\frac{-by}{a^2 - b^2}\right)^{2/3} = 1$$

$$\left(\frac{ax}{a^2 - b^2}\right)^{\frac{2}{3}} + \left(\frac{by}{a^2 - b^2}\right)^{2/3} = 1$$

The locus of $(x\,, y)$ is the four cusped hypocyloid.

$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$

**Example 4:**

Show that the evolute of the cycloid $x = a(\theta - \sin\theta)$ , $y = a(1 - cos\theta)$ is another cycloid.

**Solution:**

Given that $x = a(\theta - \sin\theta)$ , $y = a(1 - cos\theta)$

Differentiating with respect to $x$

$$\frac{dx}{d\theta} = a(1 - cos\theta) \frac{dy}{d\theta} = a(0 + sin\theta)$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{d\theta}{dx} = \frac{a \; sin\theta}{a(1 - cos\theta)}$$

$$= \frac{sin\theta}{(1 - cos\theta)}$$

$$= \frac{2sin\frac{\theta}{2} \; cos\frac{\theta}{2}}{2 \; sin^2 \theta}$$

$$= \frac{cos\frac{\theta}{2}}{sin\frac{\theta}{2}}$$

$$\frac{dy}{dx} = cot\frac{\theta}{2}$$

Again differentiating with respect to $x$

$$\frac{d^2 y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$= \frac{d}{d\theta}(cot\frac{\theta}{2})\frac{d\theta}{dx}$$

$$= -cosec^2\frac{\theta}{2} \times \frac{1}{2} \times \frac{1}{(1-cos\theta)}$$

$$= -\frac{1}{4a\ sin^4\frac{\theta}{2}}$$

$$\Rightarrow X = x - \frac{y_1(1+{y_1}^2)}{y_2}$$

$$= a(\theta - \sin\theta) + \frac{cot\frac{\theta}{2}(1+cot^2\frac{\theta}{2})}{\frac{1}{4a\ sin^4\frac{\theta}{2}}}$$

$$= a(\theta - \sin\theta) + 2a\ sin\theta$$

$$\Rightarrow X = a(\theta + \sin\theta)$$

$$\Rightarrow Y = y + \frac{(1+{y_1}^2)}{y_2}$$

$$= a(1 - cos\theta) + \frac{(1+cot^2\frac{\theta}{2})}{-\frac{1}{4a\ sin^4\frac{\theta}{2}}}$$

$$= a(1 - \cos\theta) - 2a\ (1 - \cos\theta)$$

$$\Rightarrow Y = -a(1 - cos\ \theta)$$

The locus of $(x, y)$ is,

$X = a(\theta + \sin\theta)$ , $Y = -a(1 - cos\ \theta)$

This is also a cycloid.

**Exercise 3:**

1. Find the coordinates of the centre of curvature at given points on the curves:

   (a) $y = x^2$ ; $\left(\frac{1}{2}, \frac{1}{4}\right)$

   (b) $xy = c^2$; (c,c)

   (c) $y = \log \sec x$ ; $\left(\frac{\pi}{3}, log2\right)$

2.Show that the evolute of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$$

3.Show that for the curve $x^{2/3} + y^{2/3} = a^{2/3}$ ,

$$X = a\, cos^3 t + 3a\, cost\, sin^2 t\ ,\ Y = a\, sin^3 t + 3a\, sint\, cos^2 t.$$

### 5.7. Evolute and Involute

We have already defined evolute of a curve as the locus of the centre of curvature and deduced the equations of the evolute of the parabola and ellipse

If the evolute itself be regarded as the original curve, a curve of which it is the evolute is called an involute

If may be noted that there is but one evolute but an infinite number of involutes.

### 5.8. Radius of curvature of Polar curve $r = f(\theta)$

$$\rho = \frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2 r}{d\theta^2}} = \frac{\{r^2 + {r_1}^2\}^{3/2}}{r^2 + 2{r_1}^2 - rr_2}$$

Where $r_1 = \frac{dr}{d\theta}$ , $r_2 = \frac{d^2 r}{d\theta^2}$

**Example 1:**

Find the radius of curvature of the cardioid $r = a(1 - cos\,\theta)$.

**Solution:**

Given that $r = a(1 - cos\,\theta)$

Differentiating with respect to $\theta$

$$\frac{dr}{d\theta} = a\ sin\theta$$

Again differentiating with respect to $\theta$

$$\frac{d^2 r}{d\theta^2} = a\ cos\theta$$

$$\rho = \frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2 r}{d\theta^2}}$$

$$= \frac{\{r^2 + a^2 sin^2\theta\}^{3/2}}{r^2 + 2a^2 sin^2\theta - ra\ cos\theta}$$

$$= \frac{\{a^2(1-\cos\theta)^2 + a^2 sin^2\theta\}^{3/2}}{a^2(1-\cos\theta)^2 + 2a^2 sin^2\theta - a(1-cos\ \theta)a\ cos\theta}$$

Numerator $\Rightarrow \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{3/2} = [a^2(1-\cos\theta)^2 + a^2 sin^2\theta]^{3/2}$

$= [a^2(1 + cos^2\theta - 2\ cos\theta) + a^2 sin^2\theta]^{3/2}$

$= [a^2 + a^2 cos^2\theta - 2\ a^2 cos\theta) + a^2 sin^2\theta]^{3/2}$

$= [a^2 - 2\ a^2 cos\theta) + a^2(cos^2\theta + sin^2\theta)]^{3/2}$

$= [a^2 - 2\ a^2 cos\theta) + a^2]^{3/2}$

$= [2a^2 - 2\ a^2 cos\theta)]^{3/2}$

$= [2a^2(1 - cos\theta)]^{3/2}$

$= [2a(a(1 - cos\theta))]^{3/2}$

$= [2ar]^{\frac{3}{2}}$

Denamurator$\Rightarrow r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}=a^2(1-\cos\theta)^2+2a^2sin^2\theta-a^2cos\theta\,(1-cos\theta)$

$=a^2(1+cos^2\theta-2\,cos\theta)+2a^2sin^2\theta-a^2cos\theta\,(1-cos\theta)$

$=a^2+a^2cos^2\theta-2\,a^2cos\theta)+2a^2(1-cos^2\theta)-a^2(cos\theta\;-cos^2\theta)$

$=a^2+a^2cos^2\theta-2\,a^2cos\theta+2a^2(1-cos^2\theta)-a^2cos\theta-a^2cos^2\theta$

$=3a^2-3a^2cos\theta$

$=3a^2(1-cos\theta)$

$=3a\,[a(1-cos\theta)]$

$=3ar$

$$\Rightarrow \rho=\frac{(2ar)^{3/2}}{3ar}$$

$$\Rightarrow \rho=\frac{2}{3}\sqrt{2ar}$$

**Example 2:**

Show that the radius of curvature of the curve $r^n=a^n\cos n\theta$ is $\frac{a^n r^{-n+1}}{n+1}$

**Solution:**

Let $r^n=a^n\cos n\theta$

Taking logarithms on both sides

$n\;logr=n\;loga+\log(cosn\theta)$

Differentiating with respect to $\theta$, we get

$$\frac{n}{r}\frac{dr}{d\theta}=-\frac{n\sin n\theta}{\cos n\theta}$$

$$\frac{dr}{d\theta} = -r \tan n\,\theta$$

Differentiating once again with respect to $\theta$, we get

$$\frac{d^2 r}{d\theta^2} = \frac{d}{d\theta}\left(\frac{dr}{d\theta}\right)$$

$$= \frac{d}{d\theta}(-r \tan n\,\theta)$$

$$= -\frac{dr}{d\theta}\tan n\,\theta - nrsec^2 \mathrm{n\theta}$$

$$= -(-r \tan n\,\theta)\tan n\,\theta - nrsec^2 \mathrm{n\theta}$$

$$\frac{d^2 r}{d\theta^2} = r\,tan^2\, n\,\theta - nrsec^2 \mathrm{n\theta}$$

$$\rho = \frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2 r}{d\theta^2}}$$

$$\rho = \frac{\{r^2 + (-r \tan n\,\theta)^2\}^{3/2}}{r^2 + 2(-r \tan n\,\theta)^2 - r(r\,tan^2\, n\,\theta - nrsec^2 \mathrm{n\theta})}$$

$$\rho = \frac{\{r^2 + \ r^2\, tan^2\, n\,\theta\}^{3/2}}{r^2 + 2r^2\, tan^2\, n\,\theta - r^2\, tan^2\, n\,\theta + nr^2 sec^2 \mathrm{n\theta}}$$

$$= \frac{r^3\ sec^3 \mathrm{n\theta}}{(n+1)r^2 sec^2 \mathrm{n\theta}}$$

$$= \frac{r}{(n+1)cosn\mathrm{\theta}}$$

$$= \frac{r\ a^n}{(n+1)r^n}$$

$$\Rightarrow \rho = \frac{a^n r^{-n+1}}{n+1}$$

**Exercise 4:**

1. Show that radius of curvature of the curve $r^2 = a^2 \cos 2\theta$ is $\frac{a^2}{3r}$

2.Find the radius of curvature at $(r, \theta)$ on the curve $r^n = a^n \sin n\theta$.

3.Show that in the cardioid $r = a(1 + cos\,\theta)$ , $\frac{\rho^2}{r}$ is constant.

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